

A TRANSFORMATION BETWEEN INSTITUTIONS REPRESENTING THE THEOREM OF HERBRAND-SCHMIDT-WANG

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ABSTRACT. We prove that domain unification, when viewed as a suitable transformation between two convenient institutions, represents the theorem of Herbrand-Schmidt-Wang encoding many-sorted first-order logic into single-sorted first-order logic.

1. INTRODUCTION

Before describing the theorem of Herbrand-Schmidt-Wang (first informally and then under the formal shape in which this theorem will be used in this article), we notice that from now on we tacitly assume that the terms “heterogeneous” and “many-sorted”, on the one hand, and the terms “homogeneous” and “single-sorted”, on the other hand, are synonymous.

As it is well-known, the theorem of Herbrand-Schmidt-Wang (see [9], [15], and [16]) about the reduction of heterogeneous first-order logic to homogeneous first-order logic, states that, for a heterogeneous first-order signature \mathcal{S} and a heterogeneous \mathcal{S} -theory T , given a sentence of T and a proof for it in T , there is an effective way of finding a proof in T' , the homogenization of T , for its translation in T' ; and conversely, given a sentence of T' which has a translation in T , and given a proof for it in T' , there is an effective way of finding a proof in T for its translation in T . However, the concrete and explicit formulation of the theorem of Herbrand-Schmidt-Wang, which has been extracted from [14], pp. 483–485, and which will be used in the last section of this article, is the following:

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- (1) To every heterogeneous first-order signature \mathcal{S} (see Section 2 for the definition of this notion) we associate, in a natural way, a homogeneous signature $\text{DU}(\mathcal{S})$ (see Section 4). Moreover, for each heterogeneous first-order signature \mathcal{S} , there is a syntactical translation $\alpha_{\mathcal{S}}$ (see Section 4 for its definition) taking each heterogeneous formula φ of \mathcal{S} into a homogeneous formula $\alpha_{\mathcal{S}}(\varphi)$ of the associated homogeneous signature $\text{DU}(\mathcal{S})$.
- (2) There is, for every heterogeneous first-order signature \mathcal{S} , a semantical translation $\beta_{\mathcal{S}}$ (see Section 4 for its definition), taking each pointed heterogeneous \mathcal{S} -algebraic system \mathbf{A} (see Section 2 for its definition) into a homogeneous $\text{DU}(\mathcal{S})$ -algebraic system $\beta_{\mathcal{S}}(\mathbf{A})$. Moreover, for each pointed heterogeneous \mathcal{S} -algebraic system \mathbf{A} , $\beta_{\mathcal{S}}(\mathbf{A})$ is a model of $\Phi(\mathcal{S})$, where $\Phi(\mathcal{S})$ is the set of all of the following homogeneous $\text{DU}(\mathcal{S})$ -sentences:
 - (a) $\exists v \pi_s(v)$, for each $s \in S$, where S is the underlying set of sorts of the heterogeneous first-order signature \mathcal{S} ;
 - (b) $\forall v_0, \dots, v_{n-1} (\bigwedge_{i \in n} \pi_{w_i}(v_i) \rightarrow \pi_s(\sigma(v_0, \dots, v_{n-1})))$, for each heterogeneous operation symbol σ of biarity $((w_i)_{i \in n}, s)$.
 Furthermore, let $x \in \prod_{s \in S} A_s^{\mathbb{N}}$. Then we have that $\mathbf{A} \models_{\mathcal{S}}^{\text{Ht}} \varphi[x]$ iff $\beta_{\mathcal{S}}(\mathbf{A}) \models_{\text{DU}(\mathcal{S})}^{\text{Hm}} \alpha_{\mathcal{S}}(\varphi)[x]$, where $\models_{\mathcal{S}}^{\text{Ht}}$ is the heterogeneous satisfaction relation for \mathcal{S} and $\models_{\text{DU}(\mathcal{S})}^{\text{Hm}}$ the homogeneous satisfaction relation for $\text{DU}(\mathcal{S})$.

Concerning the theorem of Herbrand-Schmidt-Wang, J. Hook in [10], p. 372, said the following: “Whenever many-sorted theories are discussed in logic texts (e.g., [14], pp. 483–485), it is fashionable to observe that every many-sorted theory \mathcal{T} can be effectively replaced by an equally powerful one-sorted theory \mathcal{T}^* This observation suggests that perhaps many-sorted theories are no more useful than one-sorted theories. That this is not always the case has been pointed out previously ([4], p. 13 (for the theorems of interpolation, we add)).” In [10] Hook proves that \mathcal{U} can be interpretable in \mathcal{T} without \mathcal{U}^* being interpretable in \mathcal{T}^* . Therefore the heterogeneous theories are not an inessential variation of the homogeneous theories, at least to the extent that ensuring the consistency of a theory has not, in general, been a trivial task ever since it was highlighted by Hilbert in 1900 at the Second International Congress of Mathematicians in Paris.

Our main goal in this article is to prove that there exists a transformation between institutions, founded on the concept of domain unification, representing the theorem of Herbrand-Schmidt-Wang. To this

end we should begin by defining two convenient institutions, one of them associated with the many-sorted first-order logic and the other associated with the single-sorted first-order logic, and by defining a suitable notion of morphism (or transformation) between institutions under which falls the encoding of many-sorted first-order logic into single-sorted first-order logic. But before doing that, for the convenience of the reader, we start by reviewing the relevant material about institutions and we continue by introducing those notions and constructions on many-sorted sets, signatures, and algebras which will be used afterwards, thus making our exposition as self-contained as possible.

The theory of institutions of Goguen and Burstall, which arose within theoretical computer science, in response to the proliferation of logics in use there, is a categorical formalization of the *semantic* or *model-theoretical* aspect of the intuitive notion of “logical system”, and it has as objectives, according to Goguen and Burstall in [7]: “(1) To support as much computer science as possible independently of the underlying logical system, (2) to facilitate the transfer of results (and artifacts such as theorem provers) from one logical system to another, and (3) to permit combining a number of different logical systems”.

We recall in passing that the concept of institution, as defined in [6], has, to the best of our knowledge, two direct logical ancestors. One of them, due to S. Feferman (relatively forgotten and, apparently, unknown by computer scientists), has to do with the concept of *regular model-theoretic language*, defined in [5], pp. 155–156. The other, due to J. Barwise (known by some computer scientists), has to do with the concept of a *logic*, defined in [1], pp. 234–235.

Our next goal is to recall that Goguen and Burstall in [6], p. 229, define an *institution* as a category **Sign**, of signatures, a functor Sen from **Sign** to **Set**, giving the set of *sentences* over a given signature, a functor Mod from **Sign** to \mathbf{Cat}^{op} , giving the category of *models* of a given signature, and, for each $\Sigma \in \mathbf{Sign}$, a satisfaction relation $\models_{\Sigma} \subseteq |\mathbf{Mod}(\Sigma)| \times \text{Sen}(\Sigma)$, where $|\cdot|$ is the endofunctor of **Cat** which sends a category to the discrete category on its set of objects, such that, for each morphism $\varphi: \Sigma \longrightarrow \Sigma'$, the

Satisfaction Condition. $\mathbf{M}' \models_{\Sigma'} \varphi(e)$ iff $\varphi(\mathbf{M}') \models_{\Sigma} e$,

holds for each $\mathbf{M}' \in |\mathbf{Mod}(\Sigma')|$ and each $e \in \text{Sen}(\Sigma)$. (Let us notice the abuse of notation on the part of Goguen and Burstall about “ $\varphi(e)$ ” and “ $\varphi(\mathbf{M}')$ ” in their formulation of the **Satisfaction Condition**. They should literally be “ $\text{Sen}(\varphi)(e)$ ” and “ $\text{Mod}(\varphi)(\mathbf{M}')$ ”, respectively.)

Concerning the functor Mod from \mathbf{Sign} to \mathbf{Cat}^{op} we recall that to give such a functor is equivalent to give a functor from $\mathbf{Sign}^{\text{op}}$ to \mathbf{Cat} . This is so since to give a *contravariant* functor F from a category \mathbf{A} to another \mathbf{B} is equivalent to give a *covariant* functor from \mathbf{A}^{op} to \mathbf{B} , or to give a *covariant* functor from \mathbf{A} to \mathbf{B}^{op} (see, e.g., [12], pp. 33–35).

On the other hand, the literature investigates several types of morphisms between institutions, see, e.g., [8], each of them playing a specific role in applications. In this article the morphisms from an institution \mathbb{I} to another \mathbb{I}' (both of them understood as in [6], p. 229, but adopting, from now on, the convention that the functors Mod and Mod' will be regarded as functors from $\mathbf{Sign}^{\text{op}}$ and $\mathbf{Sign}'^{\text{op}}$ to \mathbf{Cat} , respectively) which we will use, called *transformations*, are precisely the ordered triples (F, α, β) , where F is a functor from \mathbf{Sign} to \mathbf{Sign}' , α a natural transformation from Sen to $\text{Sen}' \circ F$, or diagrammatically:

$$\begin{array}{ccc}
 & \text{Sen} & \\
 & \curvearrowright & \\
 \mathbf{Sign} & & \mathbf{Set} \\
 & \Downarrow \alpha & \\
 & \curvearrowleft & \\
 & \text{Sen}' \circ F & \\
 F \swarrow & & \searrow \text{Sen}' \\
 & \mathbf{Sign}' &
 \end{array}$$

and β a natural transformation from Mod to $\text{Mod}' \circ F^{\text{op}}$, where F^{op} is the functor from the dual of \mathbf{Sign} to the dual of \mathbf{Sign}' , or diagrammatically:

$$\begin{array}{ccc}
 & \text{Mod} & \\
 & \curvearrowright & \\
 \mathbf{Sign}^{\text{op}} & & \mathbf{Cat} \\
 & \Downarrow \beta & \\
 & \curvearrowleft & \\
 & \text{Mod}' \circ F^{\text{op}} & \\
 F^{\text{op}} \swarrow & & \searrow \text{Mod}' \\
 & \mathbf{Sign}'^{\text{op}} &
 \end{array}$$

such that the following *satisfaction condition* holds:

$$\mathbf{A} \models_{\Sigma} \varphi \text{ iff } \beta_{\Sigma}(\mathbf{A}) \models_{F(\Sigma)} \alpha_{\Sigma}(\varphi),$$

for any $\Sigma \in \mathbf{Sign}$, any $\mathbf{A} \in \mathbf{Mod}(\Sigma)$, and any $\varphi \in \mathbf{Sen}(\Sigma)$.

We now proceed to state the basic assumptions which will be needed through the article and to recall those concepts and constructions on many-sorted sets, signatures, and algebras which will be used in the following sections.

Every set we consider in this article, unless otherwise stated, will be a \mathcal{U} -small set or a \mathcal{U} -large set, i.e., an element or a subset, respectively, of a Grothendieck universe \mathcal{U} (as defined, e.g., in [12], p. 22), fixed once and for all. Besides, we agree that \mathbf{Set} denotes the category which has as set of objects \mathcal{U} and as set of morphisms the subset of \mathcal{U} of all mappings between \mathcal{U} -small sets, and that \mathbf{Cat} denotes the category of the \mathcal{U} -categories (i.e., categories \mathbf{C} such that the set of objects of \mathbf{C} is a subset of the Grothendieck universe \mathcal{U} , and the hom-sets of \mathbf{C} elements of \mathcal{U}), and functors between \mathcal{U} -categories. Moreover, we choose, once and for all, a countably infinite set of variables $V = \{v_n \mid n \in \mathbb{N}\}$.

We agree upon calling, henceforth, for a set of sorts $S \in \mathcal{U}$, the objects of the category \mathbf{Set}^S (i.e., the elements $A = (A_s)_{s \in S}$ of \mathcal{U}^S , the set of all functions from S to \mathcal{U}) *S-sorted sets*; and the morphisms of the category \mathbf{Set}^S from an *S-sorted set* A into another B (i.e., the ordered triples (A, f, B) , abbreviated to $f: A \longrightarrow B$, where f is an element of $\prod_{s \in S} \text{Hom}(A_s, B_s)$) *S-sorted mappings from A to B*.

Let $k: S \longrightarrow T$ be a mapping, then we will denote by Δ_k the functor from \mathbf{Set}^T to \mathbf{Set}^S defined as follows: its object mapping sends each *T-sorted set* A to the *S-sorted set* $A_k = (A_{k(s)})_{s \in S}$, i.e., the composite mapping $A \circ k$; its arrow mapping sends each *T-sorted mapping* $f: A \longrightarrow B$ to the *S-sorted mapping* $f_k = (f_{k(s)})_{s \in S}: A_k \longrightarrow B_k$.

The category \mathbf{MSet} , of *many-sorted sets* and *many-sorted mappings* has as objects the pairs (S, A) , where S is a set and A an *S-sorted set*, and as morphisms from (S, A) to (T, B) the pairs (k, f) , where $k: S \longrightarrow T$ and $f: A \longrightarrow B_k$.

Our next goal is to define the category \mathbf{Sig} , of many-sorted signatures. But before doing that we agree that, for a set of sorts S in \mathcal{U} , $\mathbf{Sig}(S)$ denotes the category of *S-sorted signatures* and *S-sorted signature morphisms*, i.e., the category $\mathbf{Set}^{S^* \times S}$, where S^* is the underlying set of \mathbf{S}^* , the free monoid on S . Therefore an *S-sorted signature* is a function Σ from $S^* \times S$ to \mathcal{U} which sends a pair $(w, s) \in S^* \times S$ to the set $\Sigma_{w,s}$ of the *formal operations* of *arity* w , *sort* (or *coarity*) s , and *biarity* (w, s) ; and an *S-sorted signature morphism* from Σ to Σ' is an ordered triple (Σ, d, Σ') , written as $d: \Sigma \longrightarrow \Sigma'$, where

$d = (d_{w,s})_{(w,s) \in S^* \times S} \in \prod_{(w,s) \in S^* \times S} \text{Hom}(\Sigma_{w,s}, \Sigma'_{w,s})$. Thus, for every $(w, s) \in S^* \times S$, $d_{w,s}$ is a mapping from $\Sigma_{w,s}$ to $\Sigma'_{w,s}$ which sends a formal operation σ in $\Sigma_{w,s}$ to the formal operation $d_{w,s}(\sigma)$ ($d(\sigma)$ for short) in $\Sigma'_{w,s}$.

Remark. For a set of sorts S in \mathcal{U} , to give an S -sorted signature Σ , i.e., a function from $S^* \times S$ to \mathcal{U} is, essentially, equivalent to give an ordered triple $(\bar{\Sigma}, \text{ar}, \text{car})$, where $\bar{\Sigma} \in \mathcal{U}$, ar is a mapping from $\bar{\Sigma}$ to S^* , and car a mapping from $\bar{\Sigma}$ to S . Moreover, to give an S -sorted signature morphism d from Σ to Σ' , i.e., an element of $\prod_{(w,s) \in S^* \times S} \text{Hom}(\Sigma_{w,s}, \Sigma'_{w,s})$, is, essentially, equivalent to give a mapping \bar{d} from $\bar{\Sigma}$ to $\bar{\Sigma}'$ such that $\text{ar}_{\bar{\Sigma}} = \text{ar}_{\bar{\Sigma}'} \circ \bar{d}$ and $\text{car}_{\bar{\Sigma}} = \text{car}_{\bar{\Sigma}'} \circ \bar{d}$. These facts will be used afterwards.

Remark. If the cardinality of S is 1, i.e., if S is a final set in \mathcal{U} , then, since $S^* \times S \cong \mathbb{N}$, we have that $\mathbf{Sig}(S)$ is, essentially, the category $\mathbf{Set}^{\mathbb{N}}$, of single-sorted signatures. Hence a *single-sorted signature* is a function Σ from \mathbb{N} to \mathcal{U} which sends a natural number $n \in \mathbb{N}$ to the set Σ_n of the *formal operations* of *arity* n ; and a *single-sorted signature morphism* from Σ to Σ' is an ordered triple (Σ, d, Σ') , written as $d: \Sigma \longrightarrow \Sigma'$, where $d = (d_n)_{n \in \mathbb{N}}$ is a choice function for $(\text{Hom}(\Sigma_n, \Sigma'_n))_{n \in \mathbb{N}}$, i.e., $d = (d_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \text{Hom}(\Sigma_n, \Sigma'_n)$. Thus, for every natural number $n \in \mathbb{N}$, d_n is a mapping from Σ_n to Σ'_n which sends a formal operation σ in Σ_n to the formal operation $d_n(\sigma)$ ($d(\sigma)$ for short) in Σ'_n . To this we add that to give a single-sorted signature Σ , i.e., a function from \mathbb{N} to \mathcal{U} is, essentially, equivalent to give an ordered pair $(\bar{\Sigma}, \text{ar})$, where $\bar{\Sigma} \in \mathcal{U}$ and ar is a mapping from $\bar{\Sigma}$ to \mathbb{N} . On the other hand, to give a single-sorted signature morphism d from Σ to Σ' , i.e., an element of $\prod_{n \in \mathbb{N}} \text{Hom}(\Sigma_n, \Sigma'_n)$, is, essentially, equivalent to give a mapping \bar{d} from $\bar{\Sigma}$ to $\bar{\Sigma}'$ such that $\text{ar}_{\bar{\Sigma}} = \text{ar}_{\bar{\Sigma}'} \circ \bar{d}$.

There exists a contravariant functor \mathbf{Sig} from \mathbf{Set} to \mathbf{Cat} . Its object mapping sends each set of sorts S to $\mathbf{Sig}(S) = \mathbf{Sig}(S)$; its arrow mapping sends each mapping $k: S \longrightarrow T$ to the functor $\mathbf{Sig}(k) = \Delta_{k^* \times k}$ from $\mathbf{Sig}(T)$ to $\mathbf{Sig}(S)$ which relabels T -sorted signatures into S -sorted signatures, i.e., we have that

- (1) $\mathbf{Sig}(k)$ assigns to a T -sorted signature Λ the S -sorted signature $\mathbf{Sig}(k)(\Lambda) = \Lambda_{k^* \times k} (= \Lambda \circ (k^* \times k))$, where k^* is the underlying mapping of the monoid homomorphism from \mathbf{S}^* , the free monoid on S , to \mathbf{T}^* , the free monoid on T , canonically associated with the mapping $k: S \longrightarrow T$, and

- (2) $\text{Sig}(k)$ assigns to a morphism of T -sorted signatures d from Λ to Λ' the morphism of S -sorted signatures $\text{Sig}(k)(d) = d_{k^* \times k}$ from $\Lambda_{k^* \times k}$ to $\Lambda'_{k^* \times k}$.

The category **Sig**, of *many-sorted signatures* and *many-sorted signature morphisms* has as objects the pairs (S, Σ) , where S is a set of sorts and Σ an S -sorted signature and as many-sorted signature morphisms from (S, Σ) to (T, Λ) the pairs (k, d) , where $k: S \rightarrow T$ is a morphism in **Set** while $d: \Sigma \rightarrow \Lambda_{k^* \times k}$ is a morphism in **Sig**(S) such that the following diagrams commute

$$\begin{array}{ccc} \bar{\Sigma} = \prod_{(w,s) \in S^* \times S} \Sigma_{w,s} & \xrightarrow{\prod_{(w,s) \in S^* \times S} d_{w,s} = \bar{d}} & \prod_{(w,s) \in S^* \times S} \Lambda_{k^*(w), k(s)} = \bar{\Lambda} \\ \text{ar}_\Sigma \downarrow & & \downarrow \text{ar}_\Lambda \\ S^* & \xrightarrow{k^*} & T^* \end{array}$$

where, for every $(w, s) \in S^* \times S$, ar_Σ sends $(\sigma, (w, s)) \in \Sigma_{w,s} \times \{(w, s)\}$ to w , and ar_Λ sends $(\lambda, (k^*(w), k(s))) \in \Lambda_{k^*(w), k(s)} \times \{(k^*(w), k(s))\}$ to $k^*(w)$,

$$\begin{array}{ccc} \bar{\Sigma} = \prod_{(w,s) \in S^* \times S} \Sigma_{w,s} & \xrightarrow{\prod_{(w,s) \in S^* \times S} d_{w,s} = \bar{d}} & \prod_{(w,s) \in S^* \times S} \Lambda_{k^*(w), k(s)} = \bar{\Lambda} \\ \text{car}_\Sigma \downarrow & & \downarrow \text{car}_\Lambda \\ S & \xrightarrow{k} & T \end{array}$$

where, for every $(w, s) \in S^* \times S$, car_Σ sends $(\sigma, (w, s)) \in \Sigma_{w,s} \times \{(w, s)\}$ to s , and car_Λ sends $(\lambda, (k^*(w), k(s))) \in \Lambda_{k^*(w), k(s)}$ to $k(s)$.

The composition of two many-sorted signature morphisms (k, d) from (S, Σ) to (T, Λ) and (ℓ, e) from (T, Λ) to (U, Ω) , denoted by $(\ell, e) \circ (k, d)$, is $(\ell \circ k, e_{k^* \times k} \circ d)$, where $e_{k^* \times k}: \Lambda_{k^* \times k} \rightarrow (\Omega_{\ell^* \times \ell})_{k^* \times k} (= \Omega_{(\ell \circ k)^* \times (\ell \circ k)})$. Henceforth, unless otherwise stated, we will write Σ instead of (S, Σ) and \mathbf{d} instead of (k, d) .

Since it will be used afterwards we introduce, for a many-sorted signature Σ , an S -sorted set A , an S -sorted mapping f from A to B , and a word w on S , i.e., an element w of S^* , the following notation and terminology. We write $|w|$ for the length of the word w , A_w for $\prod_{i \in |w|} A_{w_i}$, and f_w for the mapping $\prod_{i \in |w|} f_{w_i}$ from A_w to B_w which

sends $(a_i)_{i \in |w|}$ in A_w to $(f_{w_i}(a_i))_{i \in |w|}$ in B_w . Moreover, we let $\mathcal{O}_S(A)$ stand for the $S^* \times S$ -sorted set $(\text{Hom}(A_w, A_s))_{(w,s) \in S^* \times S}$ and we call it the $S^* \times S$ -sorted set of the *finitary operations on the S -sorted set A* .

We next proceed to define the category **Alg** of many-sorted algebras. But before doing that we agree that, for a many-sorted signature Σ , **Alg**(Σ) denotes the category of Σ -algebras (and Σ -homomorphisms). By a Σ -algebra is meant a pair $\mathbf{A} = (A, F)$, where A is an S -sorted set and F a Σ -algebra structure on A , i.e., a morphism $F = (F_{w,s})_{(w,s) \in S^* \times S}$ in **Sig**(S) from Σ to $\mathcal{O}_S(A)$ (for a pair $(w, s) \in S^* \times S$ and a $\sigma \in \Sigma_{w,s}$, to simplify notation we let F_σ stand for $F_{w,s}(\sigma)$). A Σ -homomorphism from a Σ -algebra \mathbf{A} to another $\mathbf{B} = (B, G)$, is a triple $(\mathbf{A}, f, \mathbf{B})$, written as $f: \mathbf{A} \longrightarrow \mathbf{B}$, where f is an S -sorted mapping from A to B that preserves the structure, i.e., such that, for every (w, s) in $S^* \times S$, every σ in $\Sigma_{w,s}$, and every $(a_i)_{i \in |w|}$ in A_w , it happens that

$$f_s(F_\sigma((a_i)_{i \in |w|})) = G_\sigma(f_w((a_i)_{i \in |w|})).$$

Remark. For a single-sorted signature Σ , **Alg**(Σ) denotes the category of Σ -algebras (and Σ -homomorphisms). By a Σ -algebra is meant a pair $\mathbf{A} = (A, F)$, where A is a set and F a Σ -algebra structure on A , i.e., a morphism $F = (F_n)_{n \in \mathbb{N}}$ in **Set** $^{\mathbb{N}}$ from Σ to $\mathcal{O}(A)$, the family of the *finitary operations on the set A* , which is $\mathcal{O}(A) = (\text{Hom}(A^n, A))_{n \in \mathbb{N}}$ (for an $n \in \mathbb{N}$ and a $\sigma \in \Sigma_n$, to simplify notation we let F_σ stand for $F_n(\sigma)$). A Σ -homomorphism from a single-sorted algebra \mathbf{A} to another $\mathbf{B} = (B, G)$, is a triple $(\mathbf{A}, f, \mathbf{B})$, written as $f: \mathbf{A} \longrightarrow \mathbf{B}$, where f is a mapping from A to B that preserves the structure, i.e., such that, for every n in \mathbb{N} , every σ in Σ_n , and every $(a_i)_{i \in n}$ in A^n , it happens that

$$f(F_\sigma((a_i)_{i \in n})) = G_\sigma(f^n((a_i)_{i \in n})).$$

There exists a contravariant functor **Alg** from the category **Sig**, of many-sorted signatures, to **Cat**. Its object mapping sends each many-sorted signature $\Sigma = (S, \Sigma)$ to **Alg**(Σ) = **Alg**(Σ), the category of Σ -algebras; its arrow mapping sends each many-sorted signature morphism $\mathbf{d} = (k, d): \Sigma \longrightarrow \Lambda$ to the functor **Alg**(\mathbf{d}) = \mathbf{d}^* from **Alg**(Λ) to **Alg**(Σ) defined as follows: its object mapping sends each Λ -algebra $\mathbf{B} = (B, G)$ to the Σ -algebra $\mathbf{d}^*(\mathbf{B}) = (B_k, G^{\mathbf{d}})$, where $G^{\mathbf{d}}$ is the composition of the $S^* \times S$ -sorted mappings d from Σ to $\Lambda_{k^* \times k}$ and $G_{k^* \times k}$ from $\Lambda_{k^* \times k}$ to $\mathcal{O}_T(B)_{k^* \times k}$ (for $\sigma \in \Sigma_{w,s}$, to shorten notation, we let $G_{d(\sigma)}$ stand for the value of $G^{\mathbf{d}}$ at σ); its arrow mapping sends each Λ -homomorphism f from \mathbf{B} to \mathbf{B}' to the Σ -homomorphism $\mathbf{d}^*(f) = f_k$ from $\mathbf{d}^*(\mathbf{B})$ to $\mathbf{d}^*(\mathbf{B}')$.

The category **Alg**, of *many-sorted algebras* and *many-sorted algebra homomorphisms* has as objects the pairs (Σ, \mathbf{A}) , where Σ is a many-sorted signature and \mathbf{A} a Σ -algebra, and as morphisms from (Σ, \mathbf{A}) to (Λ, \mathbf{B}) , the pairs (\mathbf{d}, f) , with \mathbf{d} a many-sorted signature morphism from Σ to Λ and f a Σ -homomorphism from \mathbf{A} to $\mathbf{d}^*(\mathbf{B})$.

Moreover, we agree on the following notation and terminology. For a many-sorted signature Σ and an S -sorted set of variables X , $\mathbf{T}_\Sigma(X)$ is the *free Σ -algebra on X* , and η_X is the *insertion (of the generators) X into $\mathbf{T}_\Sigma(X)$* , the underlying S -sorted set of $\mathbf{T}_\Sigma(X)$. For a Σ -algebra \mathbf{A} and a *valuation f of the S -sorted set of variables X in \mathbf{A}* , i.e., an S -sorted mapping f from X to \mathbf{A} , we will denote by $f^\#$ the *canonical extension of f to $\mathbf{T}_\Sigma(X)$* , i.e., the unique Σ -homomorphism from $\mathbf{T}_\Sigma(X)$ to \mathbf{A} such that $f^\# \circ \eta_X = f$.

2. THE INSTITUTION ASSOCIATED WITH THE HETEROGENEOUS FIRST-ORDER LOGIC

The institution **Ht**, of *heterogeneous, or many-sorted, first-order logic*, is defined as follows. The category **Sig^{Ht}**, of *heterogeneous first-order signatures*, has as objects the quadruples $\mathcal{S} = (S, \Sigma, \Pi, j)$, where S is a set of sorts in \mathcal{U} such that $0 < \text{card}(S) \leq \aleph_0$, Σ an S -sorted signature (thus $\Sigma = (S, \Sigma)$ is a many-sorted signature), Π an S -sorted predicate domain, i.e., an object of the category $\mathbf{Set}^{S^* - \{\lambda_S\}}$, where λ_S denotes the empty word on S , and j a surjective mapping from $V (= \{v_n \mid n \in \mathbb{N}\})$ to S such that, for every $s \in S$, $j^{-1}[s]$, the fiber of j at s , is a countably infinite set; and as morphisms from $\mathcal{S} = (S, \Sigma, \Pi, j)$ to $\mathcal{S}' = (S', \Sigma', \Pi', j')$ the triples (ℓ, d, p) , where $\ell: S \rightarrow S'$ is a morphism in **Set**, $d: \Sigma \rightarrow \Sigma'_{\ell^* \times \ell}$ a morphism in **Sig**(S) (thus $\mathbf{d} = (\ell, d)$ is a many-sorted signature morphism from Σ to Σ'), and $p: \Pi \rightarrow \Pi'_{\ell^*}$ a morphism in $\mathbf{Set}^{S^* - \{\lambda_S\}}$, such that the following diagram commutes

$$\begin{array}{ccc}
 \coprod_{w \in S^* - \{\lambda_S\}} \Pi_w & \xrightarrow{\coprod_{w \in S^* - \{\lambda_S\}} p_w} & \coprod_{w \in S^* - \{\lambda_S\}} \Pi'_{\ell^*(w)} \\
 \text{rk}_\Pi \downarrow & & \downarrow \text{rk}_{\Pi'} \\
 S^* - \{\lambda_S\} & \xrightarrow{\ell^*} & S'^* - \{\lambda_{S'}\}
 \end{array}$$

where, for every $w \in S^* - \{\lambda_S\}$, rk_Π sends $(\pi, w) \in \Pi_w \times \{w\}$ to w , and $\text{rk}_{\Pi'}$ sends $(\pi', \ell^*(w)) \in \Pi'_{\ell^*(w)} \times \{\ell^*(w)\}$ to $\ell^*(w)$, $\lambda_{S'}$ is the empty

word on S' , and, by abuse of notation, we let ℓ^* stand also for the bi-restriction of ℓ^* to $S^* - \{\lambda_S\}$ and $S'^* - \{\lambda_{S'}\}$, and, in addition, ℓ satisfies the condition $\ell \circ j = j'$. Let us notice that from the equation $\ell \circ j = j'$ it follows that, for every $s \in S$, $j^{-1}[s]$ is included in $j'^{-1}[\ell(s)]$, and that ℓ is surjective.

Our next goal is to define a contravariant functor Mod^{Ht} from \mathbf{Sig}^{Ht} to \mathbf{Cat} , i.e., a covariant functor $\text{Mod}^{\text{Ht}}: (\mathbf{Sig}^{\text{Ht}})^{\text{op}} \rightarrow \mathbf{Cat}$. But before doing that we agree that, for an \mathcal{S} in \mathbf{Sig}^{Ht} , $\mathbf{Mod}^{\text{Ht}}(\mathcal{S})$ denotes the category of *pointed heterogeneous \mathcal{S} -algebraic systems* (we notice that the term “algebraic system” comes from the terminology coined by A. I. Mal'cev in [13], p. 32), i.e., the category which has as objects the quadruples $\mathbf{A} = (A, F, R, a)$, where (A, F) is a $\Sigma = (S, \Sigma)$ -algebra such that, for every sort s in S , $A_s \neq \emptyset$, R is a mapping in $\mathbf{Set}^{S^* - \{\lambda_S\}}$ from Π to $\mathcal{R}_S(A) = (\text{Sub}(A_w))_{w \in S^* - \{\lambda_S\}}$, where, for every $w \in S^* - \{\lambda_S\}$, $\text{Sub}(A_w)$ is the set of all subsets of A_w , and $a = (a_s)_{s \in S} \in \prod_{s \in S} A_s$; and as morphisms from $\mathbf{A} = (A, F, R, a)$ to $\mathbf{A}' = (A', F', R', a')$ those Σ -homomorphisms f from (A, F) to (A', F') such that, for each w in $S^* - \{\lambda_S\}$, each $\pi \in \Pi_w$, and each $x \in A_w$, if $x \in R_\pi$, then $f_w(x) \in R'_\pi$, and, for each $s \in S$, $f_s(a_s) = a'_s$. Let us denote by Mod^{Ht} the contravariant functor from \mathbf{Sig}^{Ht} to \mathbf{Cat} which sends \mathcal{S} in \mathbf{Sig}^{Ht} to $\mathbf{Mod}^{\text{Ht}}(\mathcal{S})$ and a morphism (ℓ, d, p) from \mathcal{S} to \mathcal{S}' to the functor $\text{Mod}^{\text{Ht}}(\ell, d, p)$ from $\mathbf{Mod}^{\text{Ht}}(\mathcal{S}')$ to $\mathbf{Mod}^{\text{Ht}}(\mathcal{S})$ which assigns to $\mathbf{A}' = (A', F', R', a')$ precisely $(A'_\ell, F'_{\ell^* \times \ell} \circ d, R'_{\ell^*} \circ p, (a'_{\ell(s)})_{s \in S})$. To shorten notation, for every $(w, s) \in S^* \times S$ and every $\sigma \in \Sigma_{w,s}$, we let $F'_{d(\sigma)}$ stand for the value of $F'_{\ell^* \times \ell} \circ d$ at σ , and, for every $w \in S^* - \{\lambda_S\}$ and every $\pi \in \Pi_w$, we let $R'_{p(\pi)}$ stand for the value of $R'_{\ell^*} \circ p$ at π .

We next turn to defining a functor Sen^{Ht} from \mathbf{Sig}^{Ht} to \mathbf{Set} . But before doing that we notice that, given a heterogeneous first-order signature $\mathcal{S} = (S, \Sigma, \Pi, j)$, for its underlying many-sorted signature $\Sigma = (S, \Sigma)$, we have $\mathbf{T}_\Sigma((j^{-1}[s])_{s \in S})$, the free Σ -algebra on the S -sorted set $(j^{-1}[s])_{s \in S}$. Moreover, for the logical signature $\Lambda(\mathcal{S}) = (\Lambda_n(\mathcal{S}))_{n \in \mathbb{N}}$ (which, as a matter of fact, is an example of a single-sorted signature), where $\Lambda_1(\mathcal{S}) = \{\neg\} \cup \{\forall(v, s) \mid (v, s) \in \bigcup_{s \in S} (j^{-1}[s] \times \{s\})\}$, $\Lambda_2(\mathcal{S}) = \{\wedge, \vee, \rightarrow\}$, and $\Lambda_n(\mathcal{S}) = \emptyset$, if $n \neq 1, 2$, we have $\mathbf{Fm}^{\text{Ht}}(\mathcal{S})$, the $\Lambda(\mathcal{S})$ -algebra of all heterogeneous \mathcal{S} -formulas, which is the free $\Lambda(\mathcal{S})$ -algebra on $\text{At}^{\text{Ht}}(\mathcal{S}) = \bigcup_{(w, \pi) \in \prod_{w \in S^* - \{\lambda_S\}} \Pi_w} (\{\pi\} \times \mathbf{T}_\Sigma((j^{-1}[s])_{s \in S})_w)$, the set of all heterogeneous \mathcal{S} -atomic formulas. Then let the object mapping of the functor Sen^{Ht} from \mathbf{Sig}^{Ht} to \mathbf{Set} be defined by sending \mathcal{S} in \mathbf{Sig}^{Ht}

to $\text{Sen}^{\text{Ht}}(\mathcal{S})$, the set of all heterogeneous \mathcal{S} -sentences. Our next objective is to define the morphism mapping of the functor Sen^{Ht} which, for every morphism $(\ell, d, p): \mathcal{S} \rightarrow \mathcal{S}'$, must be a mapping $\text{Sen}^{\text{Ht}}(\ell, d, p)$ from $\text{Sen}^{\text{Ht}}(\mathcal{S})$ to $\text{Sen}^{\text{Ht}}(\mathcal{S}')$. Let (ℓ, d, p) be a morphism from \mathcal{S} to \mathcal{S}' , then, by applying the functor \mathbf{d}^* from $\mathbf{Alg}(\Sigma')$ to $\mathbf{Alg}(\Sigma)$, where $\mathbf{d} = (\ell, d)$, to the free Σ' -algebra $\mathbf{T}_{\Sigma'}((j'^{-1}[s'])_{s' \in \mathcal{S}'})$, we obtain the Σ -algebra $\mathbf{d}^*(\mathbf{T}_{\Sigma'}((j'^{-1}[s'])_{s' \in \mathcal{S}'}))$. Moreover, from the natural embedding $\text{in}_{j,j'}$ of $(j^{-1}[s])_{s \in \mathcal{S}}$ into $((j'^{-1}[s'])_{s' \in \mathcal{S}'})_{\ell} = (j'^{-1}[\ell(s)])_{s \in \mathcal{S}}$, we obtain, by the universal property of the free Σ -algebra on the S -sorted set $(j^{-1}[s])_{s \in \mathcal{S}}$, the Σ -homomorphism $\text{in}_{j,j'}^{\sharp}$ from $\mathbf{T}_{\Sigma}((j^{-1}[s])_{s \in \mathcal{S}})$ to $\mathbf{d}^*(\mathbf{T}_{\Sigma'}((j'^{-1}[s'])_{s' \in \mathcal{S}'}))$. In addition, for every $w \in S^* - \{\lambda_S\}$ and every $\pi \in \Pi_w$, we have the mapping $t_{w,\pi}^{j,j'}$ from $(\mathbf{T}_{\Sigma}((j^{-1}[s])_{s \in \mathcal{S}}))_w$ to $\text{At}^{\text{Ht}}(\mathcal{S}')$ which sends $(P_i)_{i \in |w|}$ in $(\mathbf{T}_{\Sigma}((j^{-1}[s])_{s \in \mathcal{S}}))_w$ to $(p(\pi), (\text{in}_{j,j'}^{\sharp})_{w_i}(P_i))_{i \in |w|}$ in $\text{At}^{\text{Ht}}(\mathcal{S}')$. From the family of mappings $t^{j,j'} = (t_{w,\pi}^{j,j'})_{w \in S^* - \{\lambda_S\}, \pi \in \Pi_w}$ we obtain, by the universal property of the coproduct, the mapping $\text{At}^{\text{Ht}}(t^{j,j'})$ from $\text{At}^{\text{Ht}}(\mathcal{S})$ to $\text{At}^{\text{Ht}}(\mathcal{S}')$. Furthermore, the S -sorted mapping $\text{in}_{j,j'}$ from $(j^{-1}[s])_{s \in \mathcal{S}}$ to $(j'^{-1}[\ell(s)])_{s \in \mathcal{S}}$ determines a morphism $\overline{\text{in}}_{j,j'}$ from the single-sorted signature $\Lambda(\mathcal{S})$ to the single-sorted signature $\Lambda(\mathcal{S}')$. From this we conclude that there exists a forgetful functor from the category $\mathbf{Alg}(\Lambda(\mathcal{S}'))$ to the category $\mathbf{Alg}(\Lambda(\mathcal{S}))$. Let us denote by $\overline{\text{in}}_{j,j'}^*(\mathbf{Fm}^{\text{Ht}}(\mathcal{S}'))$ the value of the aforementioned functor at $\mathbf{Fm}^{\text{Ht}}(\mathcal{S}')$, the free $\Lambda(\mathcal{S}')$ -algebra on $\text{At}^{\text{Ht}}(\mathcal{S}')$. Then, by the universal property of the free $\Lambda(\mathcal{S})$ -algebra on $\text{At}^{\text{Ht}}(\mathcal{S})$, there exists a unique $\Lambda(\mathcal{S})$ -homomorphism $\text{At}^{\text{Ht}}(t^{j,j'})^{\sharp}$ from $\mathbf{Fm}^{\text{Ht}}(\mathcal{S})$ to $\overline{\text{in}}_{j,j'}^*(\mathbf{Fm}^{\text{Ht}}(\mathcal{S}'))$ such that the following diagram (in \mathbf{Set}) commutes

$$\begin{array}{ccc}
\text{At}^{\text{Ht}}(\mathcal{S}) & \xrightarrow{\eta_{\text{At}^{\text{Ht}}(\mathcal{S})}} & \mathbf{Fm}^{\text{Ht}}(\mathcal{S}) \\
\text{At}^{\text{Ht}}(t^{j,j'}) \downarrow & & \downarrow \text{At}^{\text{Ht}}(t^{j,j'})^{\sharp} \\
\text{At}^{\text{Ht}}(\mathcal{S}') & \xrightarrow{\eta_{\text{At}^{\text{Ht}}(\mathcal{S}')}} & \overline{\text{in}}_{j,j'}^*(\mathbf{Fm}^{\text{Ht}}(\mathcal{S}'))
\end{array}$$

where $\eta_{\text{At}^{\text{Ht}}(\mathcal{S})}$ is the canonical embedding of $\text{At}^{\text{Ht}}(\mathcal{S})$ into $\mathbf{Fm}^{\text{Ht}}(\mathcal{S})$, the underlying set of $\mathbf{Fm}^{\text{Ht}}(\mathcal{S})$, and $\eta_{\text{At}^{\text{Ht}}(\mathcal{S}')}$ the canonical embedding of $\text{At}^{\text{Ht}}(\mathcal{S}')$ into $\overline{\text{in}}_{j,j'}^*(\mathbf{Fm}^{\text{Ht}}(\mathcal{S}'))$, the underlying set of $\overline{\text{in}}_{j,j'}^*(\mathbf{Fm}^{\text{Ht}}(\mathcal{S}'))$. Since the direct image of $\text{Sen}^{\text{Ht}}(\mathcal{S})$ under the mapping $\text{At}^{\text{Ht}}(t^{j,j'})^{\sharp}$ is included in $\text{Sen}^{\text{Ht}}(\mathcal{S}')$, we define the mapping $\text{Sen}^{\text{Ht}}(\ell, d, p)$ from $\text{Sen}^{\text{Ht}}(\mathcal{S})$

to $\text{Sen}^{\text{Ht}}(\mathcal{S}')$ as the bi-restriction of the mapping $\text{At}^{\text{Ht}}(t^{j,j'})^\sharp$ to $\text{Sen}^{\text{Ht}}(\mathcal{S})$ and $\text{Sen}^{\text{Ht}}(\mathcal{S}')$.

Finally, let \models^{Ht} be the family $(\models_{\mathcal{S}}^{\text{Ht}})_{\mathcal{S} \in \mathbf{Sig}^{\text{Ht}}}$, where, for each $\mathcal{S} \in \mathbf{Sig}^{\text{Ht}}$, $\models_{\mathcal{S}}^{\text{Ht}}$ is the satisfaction relation associated with \mathcal{S} (see in this respect the comment of Monk in [14], p. 484).

In this way we have obtained $\mathbf{Ht} = (\mathbf{Sig}^{\text{Ht}}, \text{Mod}^{\text{Ht}}, \text{Sen}^{\text{Ht}}, \models^{\text{Ht}})$, the institution canonically associated with the heterogeneous first-order logic.

3. THE INSTITUTION ASSOCIATED WITH THE HOMOGENEOUS FIRST-ORDER LOGIC

The institution \mathbf{Hm} , of *homogeneous, or single-sorted, first-order logic*, is defined as follows. The category \mathbf{Sig}^{Hm} , of *homogeneous, or single-sorted, first-order signatures*, has as objects the ordered triples $\mathcal{S} = (\Sigma, \Pi, (\pi_s)_{s \in S})$, where Σ is a signature, i.e., an object of $\mathbf{Set}^{\mathbb{N}}$, Π a predicate domain, i.e., an object of $\mathbf{Set}^{\mathbb{N}-1}$, and $(\pi_s)_{s \in S}$ an injective mapping from S to Π_1 (the set of all unary predicate symbols), where S is a set in \mathcal{U} such that $0 < \text{card}(S) \leq \aleph_0$; and as morphisms from $\mathcal{S} = (\Sigma, \Pi, (\pi_s)_{s \in S})$ to $\mathcal{S}' = (\Sigma', \Pi', (\pi'_{s'})_{s' \in S'})$ the triples (ℓ, d, p) , where $\ell: S \rightarrow S'$ is a surjective mapping, $d = (d_n)_{n \in \mathbb{N}}: \Sigma \rightarrow \Sigma'$ a morphism in $\mathbf{Set}^{\mathbb{N}}$, and $p = (p_n)_{n \in \mathbb{N}-1}: \Pi \rightarrow \Pi'$ a morphism in $\mathbf{Set}^{\mathbb{N}-1}$, such that, for every sort $s \in S$, $p_1(\pi_s) = \pi'_{\ell(s)}$, i.e., p_1 sends, for every sort $s \in S$, in a coherent way (i.e., taking into account the surjective mapping ℓ), a distinguished predicate symbol π_s in the homogeneous first-order signature \mathcal{S} to the distinguished predicate symbol $\pi'_{\ell(s)}$ in the homogeneous first-order signature \mathcal{S}' .

Let us notice that the definition which we have provided of the concept of single-sorted first-order signature is a particular case of the following notion: A signature is an ordered quadruple $(\Sigma, \Sigma', \Pi, \Pi')$, where Σ and Π are as above, but Σ' is a subfamily of Σ , i.e., for every $n \in \mathbb{N}$, $\Sigma'_n \subseteq \Sigma_n$ and Π' is a subfamily of Π , i.e., for every $n \in \mathbb{N} - 1$, $\Pi'_n \subseteq \Pi_n$. In particular, if, for every $n \in \mathbb{N}$, $\Sigma'_n = \emptyset$ and, for every $n \in \mathbb{N} - 1$, $\Pi'_n = \emptyset$, then one obtains the ordinary first-order signatures.

The remaining components of \mathbf{Hm} , i.e., the functor Mod^{Hm} from $(\mathbf{Sig}^{\text{Hm}})^{\text{op}}$ to \mathbf{Cat} , the functor Sen^{Hm} from \mathbf{Sig}^{Hm} to \mathbf{Set} , and the family $\models^{\text{Hm}} = (\models_{\mathcal{S}}^{\text{Hm}})_{\mathcal{S} \in \mathbf{Sig}^{\text{Hm}}}$ of satisfaction relations, are defined as usual, but taking into account that, for any homogeneous first-order signature $\mathcal{S} = (\Sigma, \Pi, (\pi_s)_{s \in S})$, an homogeneous \mathcal{S} -algebraic system \mathbf{A} will be regarded as a quadruple (A, F, R, a) consisting of a Σ -algebra (A, F) , a

mapping R in $\mathbf{Set}^{\mathbb{N}-1}$ from Π to $\mathcal{R}(A) = (\text{Sub}(A^n))_{n \in \mathbb{N}-1}$, where, for every $n \in \mathbb{N} - 1$, $\text{Sub}(A^n)$ is the set of all n -ary relations on A , and a mapping a from S to A such that, for every $s \in S$, $a_s \in R_{\pi_s}(\subseteq A)$.

In this way we have obtained $\mathbf{Hm} = (\mathbf{Sig}^{\mathbf{Hm}}, \text{Mod}^{\mathbf{Hm}}, \text{Sen}^{\mathbf{Hm}}, \models^{\mathbf{Hm}})$, the institution canonically associated with the homogeneous first-order logic.

4. THE TRANSFORMATION $(\text{DU}, \alpha, \beta)$ FROM \mathbf{Ht} TO \mathbf{Hm}

We begin by defining the functor DU , of *Domain Unification*, from the category $\mathbf{Sig}^{\mathbf{Ht}}$ to the category $\mathbf{Sig}^{\mathbf{Hm}}$. Let $\mathcal{S} = (S, \Sigma, \Pi, j)$ be a heterogeneous first-order signature, then $\text{DU}(\mathcal{S})$ is the homogeneous first-order signature $(\Sigma^{\text{DU}}, \Pi^{\text{DU}}, \text{in}_S)$, where, for every $n \in \mathbb{N}$, Σ_n^{DU} , the set of all n -ary operation symbols, is $\coprod_{(w,s) \in S^n \times S} \Sigma_{w,s}$, for every $n \neq 1$, Π_n^{DU} , the set of all n -ary predicate symbols, is $\coprod_{w \in S^n} \Pi_w$, Π_1^{DU} , the set of all unary predicate symbols, is $(\coprod_{w \in S^1} \Pi_w) \coprod S$, and, finally, in_S is the canonical embedding of S into Π_1^{DU} . Let us notice that thus defined the object mapping of the functor DU takes care of the sorts, precisely by adding new unary predicate symbols: concretely those which occur in the second factor of Π_1^{DU} . The definition of the morphism mapping of DU is as follows. Let (ℓ, d, p) be a morphism from $\mathcal{S} = (S, \Sigma, \Pi, j)$ to $\mathcal{S}' = (S', \Sigma', \Pi', j')$. Then $\text{DU}(\ell, d, p)$ is the morphism from $\text{DU}(\mathcal{S}) = (\Sigma^{\text{DU}}, \Pi^{\text{DU}}, \text{in}_S)$ to $\text{DU}(\mathcal{S}') = (\Sigma'^{\text{DU}}, \Pi'^{\text{DU}}, \text{in}_{S'})$ which has as morphism from S to S' precisely to ℓ , as morphism from Σ^{DU} to Σ'^{DU} the family of mappings $(\coprod_{(w,s) \in S^n \times S} d_{w,s})_{n \in \mathbb{N}}$, and as morphism from Π^{DU} to Π'^{DU} the family of mappings defined, for $n = 1$, as $(\coprod_{w \in S^1} p_w) \coprod \ell$, and, for $n \in \mathbb{N} - 1$, as $\coprod_{w \in S^n} p_w$. It follows immediately that thus defined DU is a functor from the category $\mathbf{Sig}^{\mathbf{Ht}}$ to the category $\mathbf{Sig}^{\mathbf{Hm}}$.

Following this we define, making use of the theorem of Herbrand-Schmidt-Wang, a natural transformation β from the functor $\text{Mod}^{\mathbf{Ht}}$ to the functor $\text{Mod}^{\mathbf{Hm}} \circ \text{DU}^{\text{op}}$. Let \mathcal{S} be a heterogeneous first-order signature, then $\beta_{\mathcal{S}}$ is the functor from $\mathbf{Mod}^{\mathbf{Ht}}(\mathcal{S})$ to $\mathbf{Mod}^{\mathbf{Hm}}(\text{DU}(\mathcal{S}))$ which sends $\mathbf{A} = (A, F, R, a)$ to $(\coprod A, F^*, R^*, a^*)$, where, for every $(w, s) \in S^* \times S$ and every $\sigma \in \Sigma_{w,s}$, $F_{((w,s),\sigma)}^*$ is the mapping from $(\coprod A)^{|w|}$ to $\coprod A$ which sends $(a_i, w_i)_{i \in |w|}$ in $(\coprod A)^{|w|}$ to $(F_{\sigma}((a_i)_{i \in |w|}), s)$ in $\coprod A$, if, for every $i \in |w|$, $a_i \in A_{w_i}$, and sends $(a_i, w_i)_{i \in |w|}$ to (a_s, s) , otherwise; for every $w \in S^* - S^1$ and every $\pi \in \Pi_w$, $R_{(\pi,w)}^*$ is the subset of $(\coprod A)^{|w|}$ defined as follows

$$\{x \in (\coprod A)^{|w|} \mid \exists r \in R_{\pi}(\forall i \in |w|(x_i = (r_i, w_i)))\},$$

for every $w \in S^*$ such that $|w| = 1$ and every $\pi \in \Pi_w$, $R_{((\pi,w),0)}^*$ is defined as $R_{(\pi,w)}^*$, while, for every $s \in S$, $R_{(s,1)}^*$ is defined as the cartesian product $A_s \times \{s\}$; finally, a^* is $(a_s, s)_{s \in S}$. The definition of β_S on the morphisms is straightforward. In fact, if f is a morphism from $\mathbf{A} = (A, F, R, a)$ to $\mathbf{A}' = (A', F', R', a')$, then $\beta_S(f) = (\beta_S(\mathbf{A}), \coprod f, \beta_S(\mathbf{A}'))$. It is easy to check that thus defined β is a natural transformation from the functor Mod^{Ht} to the functor $\text{Mod}^{\text{Hm}} \circ \text{DU}^{\text{op}}$.

In addition, since we have a natural rule for translating heterogeneous statements into homogeneous statements, this allows us to define a natural transformation α from the functor Sen^{Ht} to the functor $\text{Sen}^{\text{Hm}} \circ \text{DU}$. Let \mathcal{S} be a heterogeneous first-order signature, then $\alpha_{\mathcal{S}}$ is the mapping from $\text{Sen}^{\text{Ht}}(\mathcal{S})$ to $\text{Sen}^{\text{Hm}}(\text{DU}(\mathcal{S}))$ which sends an heterogeneous \mathcal{S} -sentence φ to the homogeneous $\text{DU}(\mathcal{S})$ -sentence $\alpha_{\mathcal{S}}(\varphi)$ obtained from φ by substituting simultaneously, for each expression of the form $\forall(v, s)$ in φ an expression of the form $\forall v(\pi_s(v) \rightarrow \alpha_{\mathcal{S}}(\))$ (with the understanding that different variables in φ are replaced by different variables in $\alpha_{\mathcal{S}}(\varphi)$).

Also, by the theorem of Herbrand-Schmidt-Wang, we have that, for every heterogeneous first-order signature \mathcal{S} , every pointed heterogeneous \mathcal{S} -algebraic system \mathbf{A} , and every heterogeneous \mathcal{S} -sentence φ , $\mathbf{A} \models_{\mathcal{S}}^{\text{Ht}} \varphi$ if and only if $\beta_S(\mathbf{A}) \models_{\text{DU}(\mathcal{S})}^{\text{Hm}} \alpha_{\mathcal{S}}(\varphi)$ (see, e.g., Proposition 29.28, p. 485 in [14]). Moreover, also by Proposition 29.28, p. 485 in [14], $\beta_S(\mathbf{A})$ is a model of the set of homogeneous $\text{DU}(\mathcal{S})$ -sentences $\Phi(\mathcal{S})$, which, we recall, was defined in the first section of this article.

Therefore we have obtained a transformation between institutions $(\text{DU}, \alpha, \beta)$ from the institution Ht to the institution Hm .

It seems appropriate to mention that, concerning the functor DU , the theorem of Herbrand-Schmidt-Wang obstructs the existence of a natural transformation from $\text{Mod}^{\text{Hm}} \circ \text{DU}^{\text{op}}$ to Mod^{Ht} . The essential reason for this obstruction is that it is not always possible to associate, in a natural way, a heterogeneous algebraic system to a homogeneous algebraic system. However, by the same theorem, for every heterogeneous first-order signature \mathcal{S} , it is possible to obtain a *partial* functor $\beta_{\mathcal{S}}^{\S}$ from $\mathbf{Mod}^{\text{Hm}}(\text{DU}(\mathcal{S}))$ to $\mathbf{Mod}^{\text{Ht}}(\mathcal{S})$ in such a way that the family $\beta^{\S} = (\beta_{\mathcal{S}}^{\S})_{\mathcal{S} \in \text{Sig}^{\text{Ht}}}$ is natural in \mathcal{S} . Moreover, for each heterogeneous first-order signature \mathcal{S} , each model \mathbf{B} of $\Phi(\mathcal{S})$, and each heterogeneous \mathcal{S} -sentence φ , $\beta_{\mathcal{S}}^{\S}(\mathbf{B}) \models_{\mathcal{S}}^{\text{Ht}} \varphi$ if and only if $\mathbf{B} \models_{\text{DU}(\mathcal{S})}^{\text{Hm}} \alpha_{\mathcal{S}}(\varphi)$.

For a fuller treatment of this topic we refer the reader to [2] (we notice that the first author of this article also defined in [2], inter alia, a certain

type of 2-cell among institution morphisms from which he obtained a 2-category of institutions and a 2-functor from this 2-category to another 2-category, with 0-cells the categories of theories canonically associated with the institutions).

Remark. In connection with the problem discussed in the last paragraph of this section, i.e., the obstruction to the existence of a natural transformation from $\text{Mod}^{\text{Hm}} \circ \text{DU}^{\text{op}}$ to Mod^{Ht} , we notice that, for every heterogeneous first-order signature \mathcal{S} , the *partial* functor $\beta_{\mathcal{S}}^{\S}$ from $\mathbf{Mod}^{\text{Hm}}(\text{DU}(\mathcal{S}))$ to $\mathbf{Mod}^{\text{Ht}}(\mathcal{S})$ can, obviously, be redefined in such a way that it becomes a functor from a convenient category of algebraic systems. In fact, it suffices to consider only those algebraic systems satisfying the right set of axioms, i.e., the set $\Phi(\mathcal{S})$. More accurately, to this end one would need to define a *theoretical comorphism* (see [8], p. 292, for this notion), that translates many-sorted signatures into single-sorted *specifications*, and where sentence translation is covariant and model reduction is contravariant—in this case, the model reduction functor takes apart the underlying set of the algebraic system into several ones as prescribed by the sort predicate symbols. Then the model reduction is just an adjoint of the model translation component in our present formulation.

We should also notice, paraphrasing and in complete coincidence with what Ljapin says (in [11], p. 32) about the reduction of semigroups of *partial* transformations to semigroups of transformations, that it is not always expedient to reduce *partial* functors to functors, because, in the transition, some properties can be lost.

Remark. For future research it is an idea worth carrying out to extend the present investigation to the concept of institution defined by Goguen and Burstall in [7]. We recall that in [7] the sentence functor Sen is a **Cat**-valued functor (instead of being, simply, a **Set**-valued functor as in [6]) and where together with the **Satisfaction Condition** there is an additional axiom: the **Soundness Condition**, according to which, for every signature Σ in **Sign**, every model \mathbf{M} in $\mathbf{Mod}(\Sigma)$, and every morphism \mathcal{D} from φ to ψ in $\mathbf{Sen}(\Sigma)$, the category of Σ -sentences and Σ -morphisms (= Σ -proofs), if $\mathbf{M} \models_{\Sigma} \varphi$, then $\mathbf{M} \models_{\Sigma} \psi$.

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